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The Kuo condition, Thom's type inequality and (c)-regularity

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1 Introduction

Given a C^k mapping $f : \mathbf{R}^n \to \mathbf{R}^p$ with $f(0) = 0$, let us consider the behavior of f or the form of the zero set $f^{-1}(0)$. Even locally, they are very complicated in general. Therefore it is natural to ask when we can truncate f so that the behavior or the form of the zero-set of the truncation is similar to that of f. This problem concerns the property of sufficiency of jets. Roughly speaking, sufficiency of jets is the property that all mappings with the same truncation have the same structure.

We review some results on sufficiency of jets. Let $\mathcal{E}_{[s]}(n,p)$ denote the set of C^s map-germs : $(\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$, let $j^r f(0)$ denote the r-jet of f at $0 \in \mathbf{R}^n$ for $f \in \mathcal{E}_{[s]}(n,p)$, and let $J^r(n,p)$ denote the set of r-jets in $\mathcal{E}_{[s]}(n,p)$ $(s \geq r)$. We say $f,g \in \mathcal{E}_{[s]}(n,p)$ are C^0-equivalent, if there is a local homeomorphism $\sigma : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ such that $f = g \circ \sigma$. We further say $f, g \in \mathcal{E}_{[s]}(n,p)$ are V-equivalent (resp. SV-equivalent), if $f^{-1}(0)$ is homeomorphic to $g^{-1}(0)$ as germs at $0 \in \mathbf{R}^n$ (resp. there is a local homeomorphism $\sigma : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ such that $\sigma(f^{-1}(0)) = g^{-1}(0)$). We call an r-jet $w \in J^r(n,p)$ C^0-sufficient (resp. V-sufficient, SV-sufficient) in $\mathcal{E}_{[s]}(n,p)$ $(s \geq r)$, if any two maps $f, g \in \mathcal{E}_{[s]}(n,p)$ with $j^r f(0) = j^r g(0) = w$ are C^0-equivalent (resp. V-equivalent, SV equivalent). Concerning C^0-sufficiency of jets in the function case (i.e. p=1), we have

Theorem 1. 1 (N.Kuiper [Kui], T.C Kuo [Ku1], J.Bochnak-S.Lojasiewicz [BoLo])
For $f \in \mathcal{E}_{[r]}(n,1)$, the following conditions are equivalent.

(1) $w = j^r f(0)$ is C^0-sufficient in $\mathcal{E}_{[r]}(n,1)$.

(2) (The Kuiper-Kuo condition.) There are positive numbers $C, \alpha > 0$ such that

$$|grad\, f(x)| \geq C|x|^{r-1} \ for \quad |x| < \alpha.$$

Remark 1. 2 *The similar criterion for C^0-sufficiency of r-jets in $\mathcal{E}_{[r+1]}(n,1)$ to Theorem 1.1 is given in* [Ku1], [BoLo]. *This condition has $r-1$ replaced by $r - \delta$, with $\delta > 0$, in Theorem 1.1.*

At almost the same time as Kuiper and Kuo, R.Thom gave the following

Theorem 1. 3 (R.Thom [T])
Let $f \in \mathcal{E}_{[r]}(n,1)$. If (the Thom condition.) there are positive numbers $K, \beta > 0$ such that

$$\sum_{i<j} |x_i \frac{\partial f}{\partial x_j} - x_j \frac{\partial f}{\partial x_i}|^2 + |f(x)|^2 \geq K|x|^{2r} \ for \quad |x| < \beta$$

then $w = j^r f(0)$ is C^0-sufficient in $\mathcal{E}_{[r]}(n,1)$.

On the other hand, concerning V-sufficiency of jets, we have

Theorem 1. 4 (T.C.Kuo [Ku2])
For $f \in \mathcal{E}_{[r]}(n,p)$ $(n \geq p)$, the following conditions are equivalent.

(1) $w = j^r f(0)$ is V-sufficient in $\mathcal{E}_{[r]}(n,p)$.

(2) (The Kuo condition.) There are positive numbers $C, \alpha, \bar{w}$ such that

$$d(grad\, f_1(x), \ldots, grad\, f_p(x)) \geq C|x|^{r-1}$$

in $\mathcal{H}_r(f;\bar{w}) \cap \{|x| < \alpha\}$.

In Theorem 1.4, $\mathcal{H}_r(f;\bar{w})$ denotes the horn-neighbourhood of $f^{-1}(0)$,

$$H_r(f;\bar{w}) = \{x \in \mathbf{R}^n : |f(x)| \leq \bar{w}|x|^r\},$$

and

$$d(v_1, \ldots, v_p) = \min_i \{distance\, of\, v_i\, to\, V_i\}$$

where V_i is the span of the v_j's, $j \neq i$.

Remark 1. 5 *The similar criterion for V-sufficiency of r-jets in $\mathcal{E}_{[r+1]}(n,p)$ is also given in* [Ku2].

Throughout this note, we denote by $\bar{\rho} : \mathbf{R}^n \to \mathbf{R}$ the function defined by

$$\bar{\rho}(x) = x_1^2 + \ldots + x_n^2.$$

R.Thom [T] introduced the following condition for $f \in \mathcal{E}_{[s]}(n,p)$ $(n > p)$ which generalizes the Thom condition in Theorem 1.3 :

There are positive numbers $K, \beta, a > 0$ such that

$$\sum_{1 \leq i_1 < \ldots < i_{p+1} \leq n} |\frac{D(f_1, \ldots, f_p, \bar{\rho})}{D(x_{i_1}, \ldots, x_{i_{p+1}})}(x)|^2 + \sum_{i=1}^{p} f_i^2(x) \geq K|x|^a \ for \quad |x| < \beta.$$

We call this kind of inequality Thom's type inequality. He announced that a condition on Thom's type inequality implies SV-sufficiency of jets ([T]). In the mapping case (i.e $p \geq 2$), this condition is not necessarily economical, comparing to the Kuo condition in Theorem 1.4. Recently, D.J.A.Trotman and L.C.Wilson [TrWi] (see [Wi2] also) proved that V-sufficiency and SV-sufficiency are equivalent, using (t^r)-regularity in the stratification theory. Therefore, the Kuo condition is also equivalent to SV-sufficiency of jets.

In [B1] the first author introduced the notion of (c)-regularity which is weaker than Whitney (b)-regularity, and he showed that the (c)-regularity condition implies topological triviality. In this note, we give a characterization of (c)-regularity (Theorem 2.4). By using it, we can show that the Kuo condition in Theorem 1.4. implies the (c)-regularity condition (Theorems 2.7, 2.8). As a result, we get a different proof of the Trotman-Wilson's result (Corollary 2.9). In the proof of the result, Thom's type inequality takes a very important role. Apart from this, some condition on Thom's type inequality is equivalent to a similar condition on other type inequality (Theorem 2.14). By this result, we can see that the Thom condition in Theorem 1.3 is equivalent to the Kuiper-Kuo condition in the function case (Corollary 2.16). In other words, we can understand that Thom also has given the same result as the Kuiper-Kuo theorem. As another corollary of this, we get a result on Fukuda's ideal ([Fu]).

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Here we describe only the results. Details will appear elsewhere.

2 Main results

Let M be a smooth manifold, and let X, Y be smooth submanifolds of M such that $Y \subset \bar{X}$.

Definition 2. 1

(i) (Whitney (a)-regularity):

(X,Y) is (a)-regular at $y_0 \in Y$ if :
for each sequence of points $\{x_i\}$ which tends to y_0 such that the sequence of tangent spaces $\{T_{x_i}X\}$ tends in the Grassmann space of $\dim X$*-planes to some plane τ; then $T_{y_0}Y \subset \tau$.*
We say (X,Y) is (a)-regular if it is (a)-regular at any point $y_0 \in Y$.

(ii) ((c)-regularity):

Let ρ be a smooth non-negative function such that $\rho^{-1}(0) = Y$. (X,Y) is (c)-regular at $y_0 \in Y$ for the control function ρ if :
for each sequence of points $\{x_i\}$ which tends to y_0 such that the sequence of planes $\{Kerd\rho(x_i) \cap T_{x_i}X\}$ tends in the Grassmann space of $(\dim X - 1)$*-planes to some plane τ; then $T_{y_0}Y \subset \tau$.*
We say (X,Y) is (c)-regular for the control function ρ if it is (c)-regular at any point $y_0 \in Y$ for the control function ρ.

Remark 2. 2 *If (X,Y) is (c)-regular at $y_0 \in Y$ for some control function ρ then it is (a)-regular at $y_0 \in Y$.*

We suppose now that M is endowed with a riemannian metric .

Let (T_Y, π, ρ) be a smooth tubular neighbourhood for Y together with the associated projection and a smooth non-negative control function such that $\rho^{-1}(0) = Y$ and $grad\,\rho(x) \in Ker\, d\pi(x)$.

Definition 2. 3 *We say (X, Y) satisfies condition (m), if there exists some positive number $\epsilon > 0$ such that $(\pi, \rho)|_{X \cap T_Y^\epsilon} : X \cap T_Y^\epsilon \to Y \times \mathbf{R}$ is a submersion, where $T_Y^\epsilon = \{x \in T_Y | \ \ \rho(x) < \epsilon\}$.*

Then we can characterize (c)-regularity as follows:

Theorem 2. 4 *The pair (X, Y) is (c)-regular at $y_0 \in Y$ for the function ρ if and only if (X, Y) is (a)-regular at $y_0 \in Y$ and satisfies condition (m).*

Remark 2. 5 *In* [B2] *we have another characterization of (c)-regularity in terms of vector fields.*

This theorem is a useful criterion for (c)-regularity.

Example 2. 6 *Let $f_t : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ $(|t| < \epsilon)$ be a deformation of a C^r mapping $f = f_0$ with $j^r f_t(0) = j^r f(0)$. Assume that there are positive numbers $c, \alpha > 0$ such that*

$$(2.1) \qquad \sum_{1 \le i_1 < \ldots < i_{p+1} \le n} \left|\frac{D(f_1, \ldots, f_p, \bar{\rho})}{D(x_{i_1}, \ldots, x_{i_{p+1}})}(x)\right|^2 + \sum_{i=1}^{p} f_i^2(x) \ge c|x|^{2r}$$

for $|x| < \alpha$. Then there is $\beta > 0$ such that

$$(2.2) \qquad \sum_{1 \le i_1 < \ldots < i_{p+1} \le n} \left|\frac{D(f_{t,1}, \ldots, f_{t,p}, \bar{\rho})}{D(x_{i_1}, \ldots, x_{i_{p+1}})}(x)\right|^2 + \sum_{i=1}^{p} f_{t,i}^2(x) \ge \frac{c}{2}|x|^{2r}$$

for $|x| < \beta$ and $|t| < \epsilon$.

We define $F : (\mathbf{R}^n \times (-\epsilon, \epsilon), \{0\} \times (-\epsilon, \epsilon)) \to (\mathbf{R}^p, 0)$ by $F(x,t) = f_t(x)$. Set $X = F^{-1}(0) - \{0\} \times (-\epsilon, \epsilon)$ and $Y = \{0\} \times (-\epsilon, \epsilon)$. The following conditions follow from condition (2.2) :

(2.3) $|grad_{(x,t)} F_i| \ge \frac{c}{2}|x|^{r-1}$ on $X \cap \{|x| < \beta\}$ $(1 \le i \le p)$

$$(2.4) \qquad \sum_{1 \le i_1 < \ldots < i_{p+1} \le n} \left|\frac{D(F_1, \ldots, F_p, \bar{\rho})}{D(x_{i_1}, \ldots x_{i_{p+1}})}(x,t)\right|^2 \ne 0 \ \text{on}\ X \cap \{|x| < \beta\}.$$

Then (2.3) *implies (X, Y) is (a)-regular and* (2.4) *implies (X, Y) satisfies condition (m). Therefore it follows from Theorem 2.4 that (X, Y) is (c)-regular.*

Using Theorem 2.4 we can further show that Kuo condition implies (c)-regularity. Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ $(n \ge p)$ be a C^r (resp. C^{r+1}) mapping, and let J be a bounded open interval containing $[0, 1]$. For arbitrary $g \in \mathcal{E}_{[r]}(n, p)$ (resp. $\mathcal{E}_{[r+1]}(n, p)$) with $j^r f(0) = j^r g(0)$, define a C^r (resp. C^{r+1}) mapping $F : (\mathbf{R}^n \times J, \{0\} \times J) \to (\mathbf{R}^p, 0)$ by $F(x, t) = f(x) + t(g(x) - f(x))$. We remark that the Kuo condition guarantees that $F^{-1}(0) - \{0\} \times J$ is smooth around $\{0\} \times J$. Therefore $\Sigma(\mathbf{R}^n \times J) = \{\mathbf{R}^n \times J - F^{-1}(0), F^{-1}(0) - \{0\} \times J, \{0\} \times J\}$ gives a stratification of $\mathbf{R}^n \times J$ around $\{0\} \times J$.

Then we have

Theorem 2. 7 *If there are positive numbers $C, \alpha, \bar{w} > 0$ such that*

$$d(grad\, f_1(x), \ldots, grad\, f_p(x)) \geq C|x|^{r-1}$$

in $\mathcal{H}_r(f; \bar{w}) \cap \{|x| < \alpha\}$, then the stratification $\Sigma(\mathbf{R}^n \times J)$ is (c)-regular.

Theorem 2. 8 *If, for any polynomial mapping h of degree $r+1$ realizing $j^r f(0)$, there are positive numbers $C, \alpha, \bar{w}, \delta > 0$ such that*

$$d(grad\, f_1(x), \ldots, grad\, f_p(x)) \geq C|x|^{r-\delta}$$

in $\mathcal{H}_{r+1}(h; \bar{w}) \cap \{|x| < \alpha\}$, then the stratification $\Sigma(\mathbf{R}^n \times J)$ is (c)-regular.

As a corollary we get the Trotman-Wilson's Theorem ([TrWi], [Wi2]):

Corollary 2. 9 *For a given jet $z \in J^r(n,p)$ the following conditions are equivalent.*
(A) z is V-sufficient in $\mathcal{E}_{[r]}(n,p)$ (resp. $\mathcal{E}_{[r+1]}(n,p)$).
(B) z is SV-sufficient in $\mathcal{E}_{[r]}(n,p)$ (resp. $\mathcal{E}_{[r+1]}(n,p)$).

Remark 2. 10 *T.C.Kuo* [Ku2] *proves that in the analytic case the condition in Theorem 2.8 implies, the stratification $\Sigma(\mathbf{R}^n \times J)$ is Whitney (b)-regular.*

We must introduce some notion for C^r map germ $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$. Given a map $g \in \mathcal{E}_{[r]}(n,p)$ with $j^r g(0) = j^r f(0)$, let $f_t : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ denote the C^r mapping defined by $f_t(x) = f(x) + t(g(x) - f(x))$ for $t \in [0,1]$.

Definition 2. 11 *A condition $(*)$ on a C^r map f is $r-$compatible in the direction g, if f_t satisfies condition $(*)$ for any $t \in [0,1]$.*

If condition $(*)$ is $r-$compatible in any direction $g \in \mathcal{E}_{[r]}(n,p)$ with $j^r g(0) = j^r f(0)$, we simply say condition $(*)$ is $r-$compatible.

Remark 2. 12 *Let conditions $(*)$ and $(**)$ be $r-$compatible (or uniformly $r-$compatible in the sense of Example 2.13).*

If $(*)$ and $(**)$ are equivalent in the C^ω category then they are equivalent in the C^r category.

Example 2. 13 *(1) The Kuiper-Kuo condition in Theorem 1.1, the Thom condition in Theorem 1.3 and the Kuo condition in Theorem 1.4 (2.7) are $r-$compatible . Moreover, if f satisfies the Kuiper-Kuo condition (resp. the Thom condition, the Kuo condition), then we can take uniform (c_t, α_t) (resp. (K_t, β_t), $(C_t, \alpha_t, \bar{w}_t)$) independently from the parameter t. In this case we say condition $(*)$ is uniformly $r-$compatible.*

(2) The following condition for C^r map $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ is not $r-$compatible:

There are positive numbers $C, \alpha > 0$ such that

$$\sum_{1 \leq i_1 < \ldots < i_p \leq n} [\frac{D(f_1, \ldots, f_p)}{D(x_{i_1}, \ldots, x_{i_p})}(x)]^2 + \sum_{i=1}^{p} f_i^2(x) \geq C|x|^{2p(r-1)}$$

for $|x| < \alpha$.

In fact, take $f(x,y) = (x^8 - y^8, xy)$ then $[\frac{D(f_1,f_2)}{D(x,y)}(x,y)] = 8(x^8 + y^8)$. So then f satisfies the condition with $r = 5$.

Now take $g(x,y) = (0, xy)$ then $j^5 f(0) = j^5 g(0)$, but $f_1 = g$ does not satisfy this condition.

About Thom's type inequality (2.1), we have an equivalent condition on an other type of inequality.

Theorem 2. 14 *Let r be a positive integer.*

For a C^r mapping $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ $(n \geq p)$, the following conditions are equivalent.

(1) There are positive numbers $C, \alpha > 0$ such that

$$|x|^2 \sum_{1\leq i_1<\ldots<i_p\leq n} [\frac{D(f_1,\ldots,f_p)}{D(x_{i_1},\ldots,x_{i_p})}(x)]^2 + \sum_{i=1}^{p} f_i^2(x) \geq C|x|^{2r}$$

for $|x| < \alpha$.

(2) There are positive numbers $K, \beta > 0$ such that

$$\sum_{1\leq i_1<\ldots<i_{p+1}\leq n} |\frac{D(f_1,\ldots,f_p,\bar{\rho})}{D(x_{i_1},\ldots,x_{i_{p+1}})}(x)|^2 + \sum_{i=1}^{p} f_i^2(x) \geq K|x|^{2r}$$

for $|x| < \beta$.

Remark 2. 15 *Conditions (1) and (2) in Theorem 2.12 are uniformly $r-$compatible.*

Corollary 2. 16 *Let r be a positive integer.*

For a C^r function $f : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$, the following conditions are equivalent.

(1) (The Kuiper-Kuo condition.) There are positive numbers $C, \alpha > 0$ such that

$$|grad\, f(x)| \geq C|x|^{r-1} \qquad for \quad |x| < \alpha.$$

(2) (The Thom condition.) There are positive numbers $K, \beta > 0$ such that

$$\sum_{i<j} |x_i \frac{\partial f}{\partial x_j} - x_j \frac{\partial f}{\partial x_i}|^2 + |f(x)|^2 \geq K|x|^{2r} \; for \quad |x| < \beta.$$

Remark 2. 17 *Corollary 2.16 in the two variables case has been also obtained by T.C.Kuo, using his technique, Newton polygon with respect to a given arc.*

Let $g : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ be a C^r function such that $j^r g(0) = j^r f(0)$. Define $f_t : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ by $f_t(x) = f(x) + t(g(x) - f(x))$ for $t \in [0, 1]$.

The Kuiper-Kuo condition implies no coalescing of critical points of $\{f_t\}_{0\leq t\leq 1}$ in the sense of H.King [Ki] for any C^r realization g of $j^r f(0)$. On the other hand, the Thom condition implies that the Milnor radii of $\{f_t^{-1}(0)\}_{0\leq t\leq 1}$ are uniformly positive for any C^r realization g of $j^r f(0)$. Therefore it seems that the Thom condition is stronger than the Kuiper-Kuo condition on the surface. But it follows from Corollary 2.16 that Thom's result is equivalent to the Kuiper-Kuo Theorem.

3 Fukuda's ideal and finite determinacy.

Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ $(n \geq p)$ be a C^∞ mapping. We say f is finitely SV-determined (resp. finitely V-determined) if there is a positive integer k such that $j^k f(0)$ is SV-sufficient (resp. V-sufficient) in $\mathcal{E}_{[\infty]}(n,p)$. About finite SV-determinacy or finite V-determinacy, lots of characterizations have been obtained by J.Bochnak-T.C.Kuo [BoKu], H.Brodersen [Br] and L.C. Wilson [Wil] (see C.T.C. Wall [W] also). Here we describe a part of them. Let $\mathcal{E}$ denote the ring of C^∞ function germs : $(\mathbf{R}^n, 0) \to \mathbf{R}$, and let M_n denote the maximal ideal of $\mathcal{E}$. Let ${M_n}^\infty = \cap_{k=1}^\infty {M_n}^k$.

For a given $f \in \mathcal{E}_{[\infty]}(n,p)$, let $J(f)$ denote the ideal of $\mathcal{E}$ generated by $f_1, \ldots, f_p$ and the Jacobian determinants

$$\frac{D(f_1, \ldots, f_p)}{D(x_{i_1}, \ldots, x_{i_p})}(x) \qquad (1 \leq i_1 < \ldots < i_p \leq n).$$

Then we have

Theorem 3. 1 ([BoKu], [Br], [Wil])
For $f \in \mathcal{E}_{[\infty]}(n,p)$, the following conditions are equivalent.

(1) f is finitely SV-determined (or finitely V-determined).

(2) ${M_n}^\infty \subset J(f)$.

Next, for a given $f \in \mathcal{E}_{[\infty]}(n,p)$, let $I(f)$ denote the ideal of $\mathcal{E}$ generated by $f_1, \ldots, f_p$ and

$$\frac{D(f_1, \ldots, f_p, \bar{\rho})}{D(x_{i_1}, \ldots, x_{i_{p+1}})}(x) \qquad (1 \leq i_1 < \ldots < i_{p+1} \leq n).$$

We call $I(f)$ the Fukuda's ideal. In the paper [Fu], T.Fukuda introduced this ideal in the analytic category and discussed topological triviality under some conditions on this ideal. By definition, we have

Remark 3. 2 $I(f) \subset J(f)$.

Therefore we want to know how large Fukuda's ideal is. As a corollary of Theorem 2.14, we have

Corollary 3. 3 *For $f \in \mathcal{E}_{[\infty]}(n,p)$ $(n \geq p)$, the following conditions are equivalent.*

(1) f is finitely SV-determined (or finitely V-determined).

(2) ${M_n}^\infty \subset I(f)$.

We describe the proof of this corollary. By Theorem 3.1 it suffices to show the following conditions are equivalent.

(a) ${M_n}^\infty \subset J(f)$

(b) $M_n^\infty \subset I(f)$.

The implication $(b) \implies (a)$ follows immediately from Remark 3.2. Therefore we show the implication $(a) \implies (b)$. Assume $M_n^\infty \subset J(f)$. Then, by the Merrien-Tougeron Theorem [MTo] (see [BoKu] also), there are positive numbers $C, s, \gamma > 0$ such that

$$\sum_{1 \le i_1 < \ldots < i_p \le n} [\frac{D(f_1, \ldots, f_p)}{D(x_{i_1}, \ldots, x_{i_p})}(x)]^2 + \sum_{i=1}^{p} f_i^2(x) \ge C|x|^s$$

for $|x| < \gamma$. Let r be a positive integer such that $2r \ge s$. Then

$$|x|^2 \sum_{1 \le i_1 < \ldots < i_p \le n} [\frac{D(f_1, \ldots, f_p)}{D(x_{i_1}, \ldots, x_{i_p})}(x)]^2 + \sum_{i=1}^{p} f_i^2(x) \ge C|x|^{2(r+1)}$$

for $|x| < \gamma \le 1$. By Theorem 2.14, there are $K, \beta > 0$ such that

$$\sum_{1 \le i_1 < \ldots < i_{p+1} \le n} [\frac{D(f_1, \ldots, f_p, \bar{\rho})}{D(x_{i_1}, \ldots, x_{i_{p+1}})}(x)]^2 + \sum_{i=1}^{p} f_i^2(x) \ge K|x|^{2(r+1)}$$

for $|x| < \beta$. Using the Merrien-Tougeron Theorem again, we get $M_n^\infty \subset I(f)$.

Finally, we make one remark on Thom's type inequality.

Remark 3. 4 *When we consider triviality of a family of zero-sets or mappings, it often becomes important how we choose a neighbourhood whose boundary is transverse to zero-sets. In that case, Thom's type inequality in Theorem 2.14*(2) *is an effective tool to construct such neighbourhood. In fact, T.Fukui, the second author and M.Shiota* [FKS] *showed a modified Nash triviality theorem by using this inequality.*

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